Using Reliability Methods to Estimate the Prediction Uncertainty of a Catchment Hydrological Model

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Abstract The first-order reliability method (FORM) and the inverse reliability method (IRM) were used to estimate the prediction uncertainty of a catchment hydrological model. The parameters of the unsaturated and saturated zone components were assumed to be uncertain. The accuracy, stability and computational efficiency of the reliability methods were assessed by comparison against Latin hypercube sampling (LHS). The overall trends in the cumulative distribution functions describing uncertainty obtained from LHS were fundamentally matched by both reliability methods. However, both FORM and IRM experienced significant inaccuracies and calculation instability during optimisation of isolated performance targets that was often detrimental to their computational efficiency. It was found that IRM was the better performed of the two reliability methods because of additional optimisation criteria, but that the performance of both methods was affected by local minima and merit function behaviour. The LHS analysis was superior to the reliability methods for the analysis of multiple performance targets, although the reliability methods were more efficient for single value analysis.

1. INTRODUCTION

Computer models of hydrological processes provide predictive tools by adopting pragmatic compromises between physical accuracy and complexity. However, the inability to completely replicate the complexity of natural systems generates uncertainty in the results of every model application [Beven, 1989]. Further uncertainty is created by the choice of modelling approach, and the approximation of natural heterogeneity by model data and parameters [Yen and Guymon, 1990].

The quantification of prediction uncertainty allows results to be placed in context against data and parameter uncertainties. Monte Carlo simulation is a popular and robust procedure for estimating prediction uncertainty [eg. Freeze, 1975; Binley et al., 1991], but is computationally inefficient for low probability outcomes and complex models [McLaughlin and Wood, 1988]. Other procedures offering computational savings have been proposed, such as techniques using second moment statistical information [eg. Townley and Wilson, 1985; Connell, 1995], and point estimate procedures [eg. Harr, 1989; Yen and Guymon, 1990].

Reliability methods of uncertainty estimation are alternatives to Monte Carlo simulation and traditional second moment techniques that are believed to combine computational savings with good accuracy in highly variable systems. The basis of these methods is the probability of exceedance of a target value or failure state by the output of a model. A common procedure adopted for reliability method analysis is the first-order reliability method (FORM), which has been investigated in several hydrological modelling studies [Sitar et al., 1987; Melching, 1992; Jang et al., 1994]. An alternative method, called the inverse reliability method (IRM) in this paper, estimates uncertainty by finding the performance target associated with a known probability of exceedance. This approach is generally more appropriate for most hydrological studies, and has been examined by Bailey et al. [1996] for simple contaminant transport.

The aim of this paper is to apply both FORM and IRM to a model of an irrigated catchment, and investigate the performance of the techniques with respect to computational efficiency, stability and accuracy. Comparisons are made against a modified form of Monte Carlo analysis, and conclusions drawn regarding the suitability of the reliability methods for this type of modelling.

2. RELIABILITY METHODS

Model predictions are a function of the model's parameters and data. Analysis of prediction uncertainty assumes that several parameters are random variables with known statistical properties. Prediction uncertainty then arises by the propagation of parameter uncertainties through the model.

Reliability methods combine the parameter uncertainties with model performance targets to define prediction uncertainty. Analysis is conducted by evaluating the probability that a target is exceeded by model output:

$$P_F = P\left[g(\mathbf{X}) \leq 0\right] \tag{1}$$

where P_F refers to the probability of failure, $g(\mathbf{X})$ is a performance function that combines the performance target and model function, and $\mathbf{X}$ is a vector of random model parameters.

The integral of the joint probability distribution (PDF) of $\mathbf{X}$ in the region $g(\mathbf{X}) \leq 0$ exactly defines the probability of exceedance [Jang et al., 1994]. The complexity of the joint PDF in most hydrological models, and the scarcity of information on parameter behaviour, often makes direct evaluation of the integral impractical [Melching, 1992]. Reliability methods therefore approximate the integral for practical analysis of prediction uncertainty.

2.1 First-Order Reliability Method

The first-order reliability method (FORM) [Madsen et al., 1986; Sitar et al., 1987] approximates (1) through transformation of uncertain parameters to standard normal parameter space, and subsequent first-order approximation of the transformed performance function in standard space. A minimum of second moment statistical information is required, although marginal distribution information can be incorporated if available [Sitar et al., 1987].

The transformation of the random variables **X** uses the second moment information to determine equivalent uncorrelated standard normal deviates **U** with zero mean and unit variance. The general form of the transformation is given by Sitar et al. [1987]:

$$\mathbf{U} = \mathbf{L}_0^{-1} \begin{bmatrix} \Phi^{-1}\left[F_{x_1}(x_1)\right] \\ \vdots \\ \Phi^{-1}\left[F_{x_n}(x_n)\right] \end{bmatrix} \tag{2}$$

where $\mathbf{L}_0^{-1}$ is the lower triangular (Cholesky) decomposition of the correlation matrix $\mathbf{R}_0$; $F_{xi}(x_i)$ is the marginal cumulative distribution function of random variable x_i; and Φ^{-1} is the inverse of the standard normal cumulative distribution function. The elements $\rho_{0,ij}$ of $\mathbf{R}_0$ are obtained from semi-empirical relations derived by Der Kiureghian and Liu [1986] that link parameter correlations to marginal distributions.

Following transformation, FORM approximates the limit-state surface in standard space (defined such that $g(\mathbf{X}) = G(\mathbf{U}) = 0$, where $G(\mathbf{U})$ is the performance function in standard space) by a tangent hyperplane at the point on the surface that lies closest to the origin. This point is called the design point, and is denoted as $\mathbf{x}^*$ and $\mathbf{u}^*$ in original and standard spaces respectively.

The distance from the origin to the design point is equivalent to the magnitude of the reliability index, β, introduced by Hasofer and Lind [1974]. By linearising the limit state surface, the reliability index is obtained from the inner product:

$$\beta = \alpha^* \mathbf{u}^* \tag{3}$$

where α^* is the unit normal at the design point.

The probability of exceedance of a given performance function is approximated by:

$$P_f = \Phi(-\beta) \tag{4}$$

where Φ is the standard normal cumulative distribution function.

2.1.1 Determining the Design Point

The most computationally demanding aspect of FORM analysis is the identification of the design point. Mathematically, this is a constrained optimisation problem that requires minimisation of the distance from the origin subject to the point lying on the limit state surface [Jang et al., 1994]:

$$\begin{aligned} &\text{minimise } |\mathbf{u}| \\ &\text{subject to } G(\mathbf{u}) = 0 \end{aligned} \tag{5}$$

The Hasofer and Lind - Rackwitz and Fiessler (HL-RF) algorithm [Hasofer and Lind, 1974; Rackwitz and Fiessler, 1978] has been widely used to solve (5) [eg. Sitar et al., 1987; Jang et al., 1994]. The HL-RF algorithm locates the design point through an iterative procedure:

$$\mathbf{u}_{k+1} = \left[\nabla_u G(\mathbf{u}_k)\mathbf{u}_k - G(\mathbf{u}_k)\right] \frac{\nabla_u G(\mathbf{u}_k)^T}{\left|\nabla_u G(\mathbf{u}_k)\right|^2} \tag{6}$$

where the gradient vector in standard space is defined as:

$$\nabla_u G(\mathbf{u}) = \left[\frac{\partial G(\mathbf{u})}{\partial \mathbf{u}_1}, \ldots, \frac{\partial G(\mathbf{u})}{\partial \mathbf{u}_n}\right] \tag{7}$$

The convergence properties of the HL-RF algorithm were improved by the introduction of a merit function by Liu and Der Kiureghian [1991]. A simpler merit function was subsequently derived by Zhang and Der Kiureghian [1994]:

$$m(\mathbf{u}) = \frac{1}{2}|\mathbf{u}| + c|G(\mathbf{u})| \tag{8}$$

where c is a penalty parameter. This function has been used with the HL-RF algorithm in this study.

2.2 Inverse Reliability Method

The computational procedure of the inverse reliability method (IRM) [Der Kiureghian et al., 1994; Zhang and Der Kiureghian, 1994] is similar to that of FORM. Transformation of model parameters to standard normal space with (2) and linearisation of the limit state surface by (3) are used to define the performance function $g(\mathbf{X})$ for a known probability of exceedance and its equivalent reliability index, β_T. A similar technique is described by Schanz and Salhotra [1992].

The unknown performance function is considered to be a function of the uncertain model parameters **X** and an unknown deterministic parameter, θ, such that $g(\mathbf{X}) = g(\mathbf{X},\theta)$ [Zhang and Der Kiureghian, 1994]. The IRM analysis identifies both $\mathbf{u}^*$ (or $\mathbf{x}^*$) and θ during computation. The mathematical procedure is defined by the following equations:

$$|\mathbf{u}| - \beta_T = 0 \tag{9a}$$

$$\mathbf{u} + \frac{|\mathbf{u}|}{\left|\nabla_u G(\mathbf{u},\theta)\right|} \nabla_u G(\mathbf{u},\theta) = 0 \tag{9b}$$

$$G(\mathbf{u},\theta) = 0 \tag{9c}$$

2.2.1 Determining the Design Point for IRM

The identification of the design point and θ is obtained by the solution of (9) through optimisation. Zhang and Der Kiureghian [1994] describe an iterative algorithm based upon the concepts of the HL-RF algorithm for FORM:

$$\mathbf{u}_{k+1} = -\beta_T \frac{\nabla_u G(\mathbf{u}_k, \theta_k)}{|\nabla_u G(\mathbf{u}_k, \theta_k)|} \tag{10a}$$

$$\begin{aligned}\theta_{k+1} = \ & \theta_k + \frac{\langle \nabla_u G(\mathbf{u}_k, \theta_k), \mathbf{u}_k \rangle}{\partial G(\mathbf{u}_k, \theta_k)/\partial\theta} \\ & - \frac{G(\mathbf{u}_k, \theta_k)}{\partial G(\mathbf{u}_k, \theta_k)/\partial\theta} + \frac{\beta_T |\nabla_u G(\mathbf{u}_k, \theta_k)|}{\partial G(\mathbf{u}_k, \theta_k)/\partial\theta}\end{aligned} \tag{10b}$$

Zhang and Der Kiureghian [1994] use a merit function to assist the convergence of the algorithm. The appearance of the algorithm is similar to (8):

$$m(\mathbf{u}, \theta) = \frac{1}{2}|\mathbf{u}|^2 + c|G(\mathbf{u}, \theta)| \tag{11}$$

Implementation of the algorithm requires $c > \beta_T |\mathbf{u}| / \delta$, where δ is the required tolerance in satisfying (9c).

3. STUDY AREA AND MODEL

The catchment of the Drain 14 surface drainage system of the Kerang Irrigation District in northern Victoria (Figure 1) was used for assessment of FORM and IRM under application to a catchment hydrological model. This catchment has an area of 2728 ha that is predominantly irrigated pasture. Flood irrigation is widely used in the region, and is supplied by a network of supply channels. Excess irrigation and farm effluent is removed by the surface drainage system. Further details of the catchment are described in McAllister and Barrs [1994].

The Drain 14 catchment is simulated by the SMILE model [Beverly et al., 1997]. A similar modelling study of the catchment was previously conducted with the SWAGSIM model [Prathapar et al., 1995]. Structural similarities between the two models allowed much of the data collected by Prathapar et al. [1995] to be used in the current modelling study and uncertainty analysis.

SMILE is a distributed model that employs physically-based routines for the horizontal movement of water in the saturated zone, and a lumped conceptual representation of the unsaturated zone [refer Beverly et al., 1997]. The parameters of the unsaturated zone processes were used in the calibration of the model against regional bores and irrigation bay piezometric levels [Poulton and Slater, 1995].

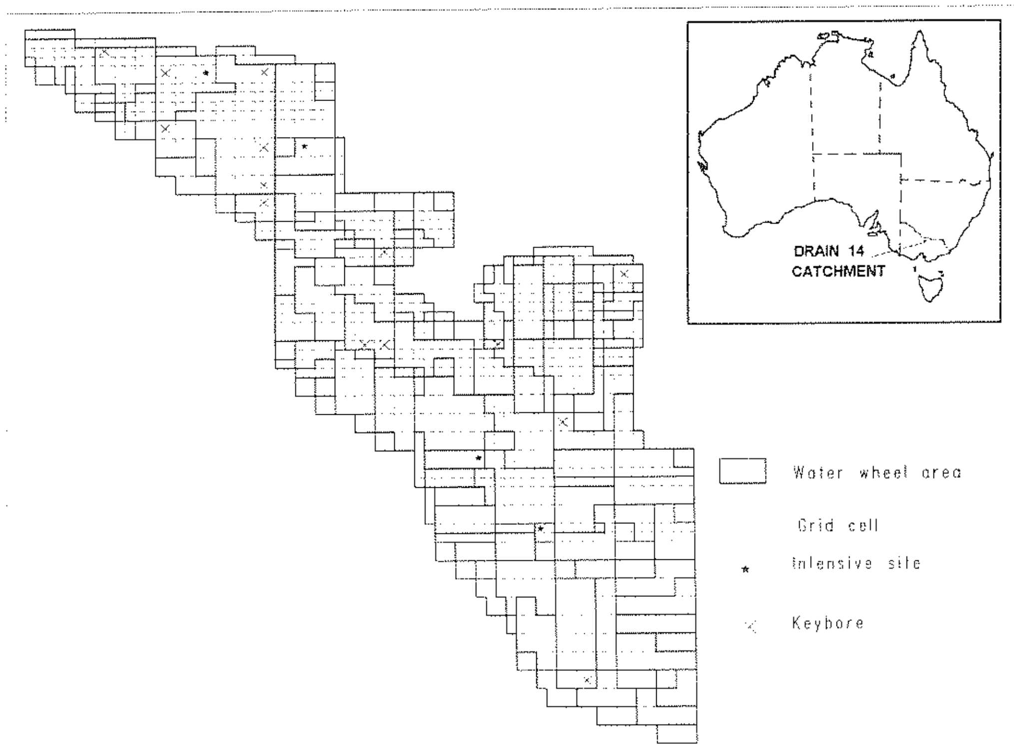

Figure 1: Drain 14 Grid, Irrigation Supply Areas and Calibration Bores used in SMILE Model [after Prathapar et al., 1995]

4. RESULTS AND DISCUSSION

Application of the reliability methods to the Drain 14 model assumed uncertainty could exist in the parameters of the unsaturated and saturated zones. Most scenarios examined the effects of uncertainty in parameters of the unsaturated zone conceptualisation because of their importance to catchment behaviour.

The analysis of uncertainty was conducted for several performance functions based on different components of catchment response predicted by the SMILE model. A series of these performance functions concentrated upon the groundwater recharge characteristics of clusters of cells within the catchment. The recharge characteristics within these clusters are indicative of the potential for salinisation within a catchment, as zones of net discharge suggest discharge from the watertable.

Both reliability methods were applied to the performance functions based on recharge characteristics. Their performance was assessed against Latin hypercube sampling (LHS) [McKay, 1988] analysis conducted for each scenario. The LHS technique is similar to Monte Carlo simulation, but uses stratified sampling of parameter PDFs to reduce the number of realisations required.

A typical assessment of FORM and IRM was conducted by assuming the recharge partitioning exponents of soils within the catchment to be uncertain, and examining the uncertainty that was developed in the predictions of net recharge to groundwater in nine finite difference cells located in the west of the Drain 14 catchment. Although only a single component of the unsaturated zone conceptualisation, sensitivity analyses had indicated that model output was sensitive to changes in this parameter. It also represented a compromise between the sources of uncertainty and the execution times required for analysis.

Each of the three recharge exponents was described by a symmetrical triangular distribution bounded by zero and twice its mean value, with the calibrated value taken as the mean. The coefficient of variation for each parameter was therefore 0.41.

The cumulative distribution function (CDF) of probability of exceedance of performance targets derived from the application of FORM were compared with the corresponding LHS analysis (Figure 2). It was observed that the CDF curves exhibit similar trends, but that the FORM analysis could deviate severely from the LHS curve. The source of this deviation was failure of the FORM analysis to converge to correct solution for the required performance target. This was mostly due to the presence of local minima in the limit state surface, which affected the convergence properties of the FORM HL-RF algorithm. The merit function exacerbated the problem on several occasions, as the identification of the local minimum caused the merit function to be continually implemented. This contributed to a significant computation burden for FORM in this particular scenario.

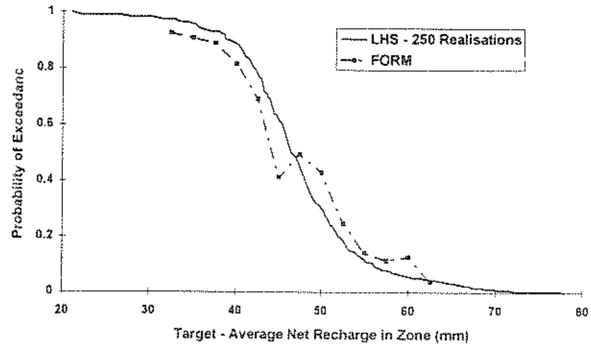

Figure 2: Comparison of Cumulative Distribution Functions Derived by FORM and LHS Analysis

An IRM analysis using the same conditions was conducted for probabilities of exceedance between 5 and 95%. As demonstrated in Figure 3, the IRM experienced similar difficulties in optimisation to the FORM analysis. The additional constraint placed upon the optimisation procedure allowed some design points and targets to be identified with reduced computational effort. However, the computational effort was still significant in comparison to the LHS analysis, and was again primarily due to the merit function modification being unable to modify the convergence properties of the modified HL-RF algorithm.

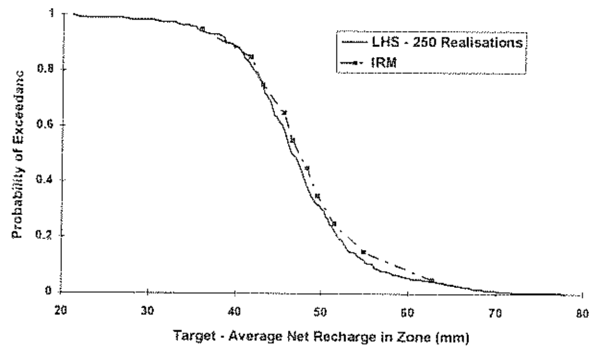

Figure 3: Comparison of Cumulative Distribution Functions Derived by IRM and LHS Analysis

The overall failures of the FORM and IRM procedures with the recharge-based performance functions was further demonstrated by poor evaluation of the probability distribution functions of the performance function (Figure 4). The calculated PDFs contain strong discontinuities because of the inaccurate optimisations, and compare poorly against the curves obtained from LHS analysis. This is clearly indicative of poor performance, as the PDFs derived from LHS analysis are themselves quite poor because of the small sample size.

The application of the reliability methods for similarly defined performance functions involving other parameters of the unsaturated and saturated zones for different marginal distributions revealed similar deficiencies. These failings were again predominantly caused by the merit function failing to assist convergence, and creating an oscillating effect in situations of poor convergence.

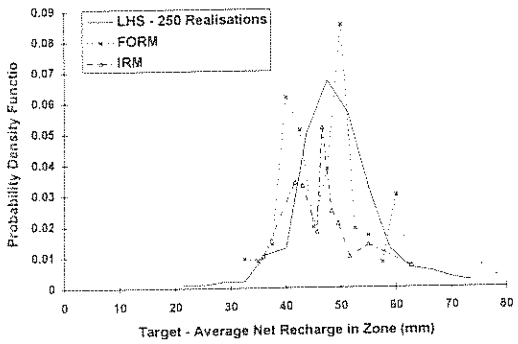

Figure 4: Comparison of Probability Distribution Functions Derived by Reliability Methods and LHS Analysis

5. CONCLUSIONS

The objective of this paper was to examine the computational requirements, accuracy and stability of reliability method applied to a catchment water balance model. Emphasis was placed on the relative abilities of FORM and IRM to estimate prediction uncertainty arising from typical usage of the catchment model.

The analysis conducted with the SMILE model of the Drain 14 catchment suggested that the two reliability methods experience difficulties that limit their effectiveness when used to define the probability and cumulative distribution functions of model output. The FORM analyses demonstrated occasional instability that severely affected the accuracy of uncertainty estimates. In addition, the sought-after computational savings thought possible with FORM were rarely achieved because of difficulty in identifying a design point, most probably the result of local minima.

Inverse reliability method analyses fared somewhat better than the FORM analysis. The greater constraint on its search algorithm provided by modification of the performance target seemed to assist convergence. However, it also struggled to achieve a satisfactory convergence on several occasions.

The comparisons of the reliability approaches against Latin hypercube simulation demonstrated the inaccuracies of their application. Latin hypercube simulation proved a more efficient and less problematic technique for the cumulative distribution function describing probability of exceedance. However, this procedure is inefficient for analysis based upon a single outcome. In such circumstances, and with greater control over calculation, reliability methods, and the inverse reliability method in particular, are more efficient.

The analyses conducted for the Drain 14 model examined a portion of the uncertainty that exist in catchment hydrological models. Uncertainty in the accuracy of rainfall and irrigation deliveries, and the discretisation of the natural environment all contribute to prediction uncertainty. Further research into the incorporation of these uncertainties into analysis of catchment hydrological models is continuing.

6. ACKNOWLEDGMENTS

Mark Bailey wishes to acknowledge the contribution of Goulburn-Murray Water in providing assistance for his studies at Monash University. Craig Beverly is acknowledged for his assistance with the SMILE model.

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